Addendum: Viscous Stable Cosserat Rods

Jerry Hsu
jerry.hsu.research@gmail.com

March, 2026

Abstract

We extend the Stable Cosserat Rods framework [1] with Kelvin–Voigt style strain rate damping on both the stretching/shearing and bending/twisting strains. The damping is formulated to avoid penalizing rigid body rotation: stretching damping operates on the local-frame strain rate, while bending damping penalizes the rate of change of relative orientation between adjacent segments. By exploiting the structural similarity to the existing energies, both extensions integrate into the solver with minimal modification: the material axis e_3 is replaced by a generalized axis $\boldsymbol{\eta}$, and the bending accumulator $\mathbf{b}$ absorbs additional viscous terms.

1 Background

We briefly review the relevant parts of the Stable Cosserat Rods formulation. A discretized Cosserat rod stores positions $\mathbf{x}_i$ on vertices and unit quaternion orientations q_i on segments. The stretching and shearing constraint on segment i is

$$C_i^{ss} = \frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{l_i} - d_{i,3}, \qquad d_{i,3} = q_i \, e_3 \, \bar{q}_i, \tag{1}$$

with rest length l_i. The bending and twisting constraint between segments i and $i+1$ is

$$C_i^{bt} = \bar{q}_i \, q_{i+1} - \phi_i \, q_i^0, \tag{2}$$

where q_i^0 is the rest relative orientation and ϕ_i selects the nearest quaternionic pole. The total elastic energy is

$$E^{total} = \sum_i \frac{k_i^{ss}}{2} |C_i^{ss}|^2 + \sum_i \frac{k_i^{bt}}{2} |C_i^{bt}|^2. \tag{3}$$

Under the quasi-static orientation assumption ($J = 0$), the orientation solve reduces to a local Gauss–Seidel relaxation. For each segment i, the equilibrium condition with Lagrange multiplier λ enforcing $|q_i| = 1$ is

$$\mathbf{v} \, q_i \, e_3 + \mathbf{b} - \lambda \, q_i = 0, \tag{4}$$

where the stretching contribution $\mathbf{v}$ and bending contribution $\mathbf{b}$ are

$$\mathbf{v} = -\frac{2k_i^{ss}}{l_i} (\mathbf{x}_{i+1} - \mathbf{x}_i), \qquad \mathbf{b} = \sum_{C^{bt} \ni i} (\text{bending terms}). \tag{5}$$

Specifically, the bending contributions to $\mathbf{b}$ from the elastic energy are

$$\mathbf{b}^{el} = k_{i-1}^{bt} \, \phi_{i-1} \, q_{i-1} \, q_{i-1}^0 + k_i^{bt} \, \phi_i \, q_{i+1} \, \overline{q_i^0}. \tag{6}$$

The closed-form solution is obtained by right-multiplying eq. (4) by e_3, using $e_3^2 = -1$ to isolate $q_i e_3$, and substituting back:

$$q_i(\lambda) = \frac{\mathbf{v}\mathbf{b}e_3 + \lambda\mathbf{b}}{\lambda^2 - |\mathbf{v}|^2}, \tag{7}$$

with approximate $\lambda \approx |\mathbf{v}| + |\mathbf{b}|$.

2 Damping Energies

2.1 Stretching/shearing strain rate damping

We define a damping potential on the rate of change of the local stretching and shearing strain. The local-space strain on segment i is

$$\Gamma_i^{loc} = \bar{q}_i \frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{l_i} q_i - e_3. \tag{8}$$

Using a backward finite difference with the previous-frame local strain $\bar{\Gamma}_i$, the stretching damping energy is

$$E_i^d = \frac{\beta_{ss}\, k_i^{ss}}{2h^2} \left|\Gamma_i^{loc} - \bar{\Gamma}_i\right|^2, \tag{9}$$

where h is the time step, $\bar{\Gamma}_i$ is treated as a constant for the current solve, and $\beta_{ss} \geq 0$ is the stretching damping parameter (units of time squared). This is stiffness-proportional: stiffer segments are damped proportionally more.

Reformulation as a world-space norm. Since quaternion rotation preserves norms, $|\bar{q}\,\mathbf{a}\,q| = |\mathbf{a}|$, we rotate the argument of the norm by q_i:

$$\left|\bar{q}_i\,\mathbf{p}\,q_i - (e_3 + \bar{\Gamma}_i)\right|^2 = \left|\mathbf{p} - q_i(e_3 + \bar{\Gamma}_i)\bar{q}_i\right|^2, \tag{10}$$

where $\mathbf{p} = (\mathbf{x}_{i+1} - \mathbf{x}_i)/l_i$. Defining the *damping rest direction*

$$\mathbf{r}_i = e_3 + \bar{\Gamma}_i, \tag{11}$$

the stretching damping energy takes the form

$$E_i^d = \frac{\beta_{ss}\, k_i^{ss}}{2h^2} \left|\frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{l_i} - q_i\,\mathbf{r}_i\,\bar{q}_i\right|^2. \tag{12}$$

This is structurally identical to the stretching/shearing energy E_i^{ss} with the replacements $e_3 \to \mathbf{r}_i$ and $k_i^{ss} \to \alpha_{ss}\, k_i^{ss}$, where $\alpha_{ss} = \beta_{ss}/h^2$.

2.2 Bending/twisting strain rate damping

We similarly penalize the rate of change of the bending/twisting strain. The relative orientation between segments i and $i+1$ is the quaternion product $\bar{q}_i\, q_{i+1}$. Using a backward finite difference with the previous-frame relative orientation $\bar{\Omega}_i = \bar{q}_i^{prev}\, q_{i+1}^{prev}$, the bending damping energy is

$$E_i^{bd} = \frac{\beta_{bt}\, k_i^{bt}}{2h^2} \left|\bar{q}_i\, q_{i+1} - \psi_i\, \bar{\Omega}_i\right|^2, \tag{13}$$

where $\beta_{bt} \geq 0$ is the bending damping parameter, and ψ_i selects the nearest quaternionic pole (same logic as ϕ_i in eq. (2), comparing against $\bar{\Omega}_i$ instead of q_i^0). This is structurally identical to E_i^{bt} with the replacements $q_i^0 \to \bar{\Omega}_i$, $\phi_i \to \psi_i$, and $k_i^{bt} \to \alpha_{bt}\, k_i^{bt}$, where $\alpha_{bt} = \beta_{bt}/h^2$.

3 Torque Derivation

3.1 Stretching damping torque

The torque on q_i from the stretching damping energy follows by the same quaternion calculus as for E^{ss}. For an energy of the form $E = \frac{k}{2}|\mathbf{p} - q\,\mathbf{a}\,\bar{q}|^2$, the quaternion gradient is

$$-\nabla_q E = -2k\,(\mathbf{p} - q\,\mathbf{a}\,\bar{q})\,q\,\mathbf{a}. \tag{14}$$

Applying this to eq. (12):

$$\tau^d_{i,i} = -\frac{2\beta_{ss}\,k^{ss}_i}{h^2}\left(\frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{l_i} - q_i\,\mathbf{r}_i\,\bar{q}_i\right) q_i\,\mathbf{r}_i. \tag{15}$$

Expansion and absorption into λ. Expanding the quaternion product and using $\mathbf{r}_i^2 = -|\mathbf{r}_i|^2$ for pure imaginary quaternions:

$$\tau^d_{i,i} = -\frac{2\beta_{ss}\,k^{ss}_i}{h^2}\left(\mathbf{p}\,q_i\,\mathbf{r}_i + |\mathbf{r}_i|^2\,q_i\right). \tag{16}$$

The term $|\mathbf{r}_i|^2\,q_i$ is proportional to q_i and is absorbed into the Lagrange multiplier λ. The remaining contribution $\mathbf{p}\,q_i\,\mathbf{r}_i$ enters the equilibrium equation.

3.2 Bending damping torque

The torque from E^{bd}_i on q_i follows by direct analogy with the elastic bending torques. By the same quaternion calculus that yields eq. (6), the bending damping energy E^{bd} contributes torques

$$\tau^{bd}_{i,i} = -\frac{\beta_{bt}\,k^{bt}_i}{h^2}\left(q_i - \psi_i\,q_{i+1}\,\overline{\bar{\Omega}_i}\right), \tag{17}$$

$$\tau^{bd}_{i-1,i} = -\frac{\beta_{bt}\,k^{bt}_{i-1}}{h^2}\left(q_i - \psi_{i-1}\,q_{i-1}\,\bar{\Omega}_{i-1}\right), \tag{18}$$

where $\tau^{bd}_{i,i}$ is the torque from E^{bd}_i on q_i (acting as the first segment in the pair), and $\tau^{bd}_{i-1,i}$ is the torque from E^{bd}_{i-1} on q_i (acting as the second segment). The terms proportional to q_i are again absorbed into λ.

4 Modified Equilibrium

4.1 Stretching damping contribution to v

As derived in section 3, the stretching damping contributes $\mathbf{p}\,q_i\,\mathbf{r}_i$ terms to the equilibrium. Since

$$\mathbf{v}_d = -\frac{2\beta_{ss}\,k^{ss}_i}{h^2\,l_i}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \alpha_{ss}\,\mathbf{v}, \qquad \alpha_{ss} = \frac{\beta_{ss}}{h^2}, \tag{19}$$

the stretching and its damping factor into a single term $\mathbf{v}\,q_i(e_3 + \alpha_{ss}\,\mathbf{r}_i)$. Define the *generalized material axis*

$$\boldsymbol{\eta}_i = e_3 + \alpha_{ss}\,\mathbf{r}_i = (1 + \alpha_{ss})\,e_3 + \alpha_{ss}\,\bar{\Gamma}_i. \tag{20}$$

Note that under stiffness-proportional damping, α_{ss} is *uniform across all segments*—it depends only on the global parameter β_{ss} and the time step, not on per-segment stiffness.

4.2 Bending damping contribution to b

After absorbing the q_i-proportional terms into λ, the remaining bending damping contributions add directly to $\mathbf{b}$. Defining $\alpha_{bt} = \beta_{bt}/h^2$, the total bending accumulator becomes

$$\mathbf{b} = \mathbf{b}^{el} + \mathbf{b}^{damp}, \tag{21}$$

where $\mathbf{b}^{el}$ is the elastic contribution from eq. (6) and

$$\mathbf{b}^{damp} = \alpha_{bt}\, k^{bt}_{i-1}\, \psi_{i-1}\, q_{i-1}\, \bar{\Omega}_{i-1} + \alpha_{bt}\, k^{bt}_{i}\, \psi_{i}\, q_{i+1}\, \overline{\bar{\Omega}_{i}}. \tag{22}$$

This generalizes naturally to arbitrary graph connectivity, mirroring the summation form in [1]:

$$\mathbf{b}_i = \sum_{C^{bt} \ni i} \begin{cases} (k + \alpha_{bt}\, k)\, \phi\, q_j\, q^0 & \text{if } \phi = \psi \text{ and } q^0 = \bar{\Omega}, \\ k\, \phi\, q_j\, q^0 + \alpha_{bt}\, k\, \psi\, q_j\, \bar{\Omega} & \text{otherwise.} \end{cases} \tag{23}$$

4.3 Combined equilibrium

The modified equilibrium is

$$\boxed{\mathbf{v}\, q_i\, \boldsymbol{\eta} + \mathbf{b} - \lambda\, q_i = 0,} \tag{24}$$

where $\boldsymbol{\eta}$ encodes the stretching damping (eq. (20)) and $\mathbf{b}$ includes both elastic and viscous bending contributions (eq. (21)). This has the identical structure to eq. (4) with $e_3 \to \boldsymbol{\eta}$ and an augmented $\mathbf{b}$.

5 Closed-Form Solution

The derivation is identical to the undamped case, since it depends only on the algebraic structure of eq. (24).

5.1 Derivation of $q(\lambda)$

Right-multiply eq. (24) by $\boldsymbol{\eta}$ and use $\boldsymbol{\eta}^2 = -|\boldsymbol{\eta}|^2$:

$$-|\boldsymbol{\eta}|^2\, \mathbf{v}\, q_i + \mathbf{b}\boldsymbol{\eta} - \lambda\, q_i \boldsymbol{\eta} = 0. \tag{25}$$

Left-multiply eq. (25) by $\mathbf{v}$ and use $\mathbf{v}^2 = -|\mathbf{v}|^2$:

$$|\boldsymbol{\eta}|^2 |\mathbf{v}|^2\, q_i + \mathbf{v}\mathbf{b}\boldsymbol{\eta} - \lambda\, \mathbf{v}\, q_i \boldsymbol{\eta} = 0. \tag{26}$$

From eq. (24), rearranging gives $\mathbf{v}\, q_i \boldsymbol{\eta} = \lambda\, q_i - \mathbf{b}$. Substituting into eq. (26):

$$|\boldsymbol{\eta}|^2 |\mathbf{v}|^2\, q_i + \mathbf{v}\mathbf{b}\boldsymbol{\eta} - \lambda(\lambda\, q_i - \mathbf{b}) = 0. \tag{27}$$

Collecting the q_i terms and solving:

$$\boxed{q_i(\lambda) = \frac{\mathbf{v}\mathbf{b}\boldsymbol{\eta} + \lambda\, \mathbf{b}}{\lambda^2 - |\mathbf{v}|^2 |\boldsymbol{\eta}|^2}.} \tag{28}$$

This is the generalized form of eq. (7). When $\boldsymbol{\eta} = e_3$ and $|\boldsymbol{\eta}| = 1$, it reduces to the original.

5.2 Approximate λ

The unit-norm constraint $|q_i(\lambda)| = 1$ and the triangle inequality give bounds

$$|\mathbf{v}||\boldsymbol{\eta}| < \lambda \leq |\mathbf{v}||\boldsymbol{\eta}| + |\mathbf{b}|, \tag{29}$$

and the approximate solution is the upper bound:

$$\boxed{\lambda \approx |\mathbf{v}||\boldsymbol{\eta}| + |\mathbf{b}|.} \tag{30}$$

5.3 Exact λ via fixed-point iteration

The fixed-point operator generalizes directly:

$$\lambda_{fp} = f(\lambda) = \sqrt{|\mathbf{v}\mathbf{b}\boldsymbol{\eta} + \lambda\mathbf{b}| + |\mathbf{v}|^2|\boldsymbol{\eta}|^2}, \tag{31}$$

with γ_i stored such that $\lambda_i = |\mathbf{v}_i||\boldsymbol{\eta}_i| + \gamma_i|\mathbf{b}_i|$.

6 Well-Definedness

Zero strain, steady state. Under zero strain ($C^{ss} = 0$, $C^{bt} = 0$) and steady state ($\bar{\Gamma} = 0$, $\bar{\Omega}_i = q_i^0$), we have $\mathbf{r} = e_3$, $\boldsymbol{\eta} = (1 + \alpha_{ss})\, e_3$, and the bending damping terms become proportional to their elastic counterparts: $\mathbf{b}^{damp} = \alpha_{bt}\, \mathbf{b}^{el}$, so $\mathbf{b} = (1 + \alpha_{bt})\, \mathbf{b}^{el}$. Substituting $\mathbf{v} = -|\mathbf{v}|\, q\, e_3\, \bar{q}$ and $\mathbf{b} = |\mathbf{b}|\, q$:

$$q(\lambda) = \frac{|\mathbf{b}|\big(|\mathbf{v}|(1 + \alpha_{ss}) + \lambda\big)}{\lambda^2 - |\mathbf{v}|^2(1 + \alpha_{ss})^2}\, q. \tag{32}$$

With $\lambda = |\mathbf{v}|(1 + \alpha_{ss}) + |\mathbf{b}|$, the prefactor reduces to 1 and $q(\lambda) = q$. The model introduces no artificial strain. ✓

Frame indifference. Rotating all quantities by an arbitrary quaternion R: $\mathbf{v} \to R\mathbf{v}\bar{R}$, $\mathbf{b} \to R\mathbf{b}$ (both elastic and damping terms transform uniformly), $\boldsymbol{\eta}$ is unchanged (it lives in local space), and $|\mathbf{v}|$, $|\mathbf{b}|$, $|\boldsymbol{\eta}|$ are all invariant. The solution $q_i(\lambda)$ transforms as $q \to Rq$. ✓

Vertex ordering invariance. The damping extensions inherit the vertex ordering convention of the underlying Cosserat rod discretization, where segments are directed from vertex i to $i+1$. Since $\bar{\Gamma}_i$ and hence $\boldsymbol{\eta}_i$ depend on this convention through $\mathbf{p} = (\mathbf{x}_{i+1} - \mathbf{x}_i)/l_i$, vertex ordering invariance of the full system follows from the base formulation rather than from the damping terms in isolation.

7 Position Forces and Hessian

The stretching damping energy contributes to the VBD position update. The force on vertex $\mathbf{x}_i$ from the stretching damping is

$$\mathbf{f}^d_{i,i} = -\nabla_{\mathbf{x}_i} E^d_i = \frac{\beta_{ss}\, k^{ss}_i}{h^2\, l_i} \left(\frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{l_i} - q_i\, \mathbf{r}_i\, \bar{q}_i \right), \tag{33}$$

with diagonal Hessian

$$\nabla_{\mathbf{x}_i} \mathbf{f}^d_{i,i} = \frac{\beta_{ss}\, k^{ss}_i}{h^2\, l_i^2}\, \mathbf{I}. \tag{34}$$

These add directly to the existing E^{ss} force and Hessian contributions in the VBD block-diagonal solve.

The bending damping energy E^{bd} depends only on quaternions and does not contribute to the position forces.

8 Implementation Summary

The changes to Algorithm 2 of [1] are:

Start of each frame (before the iterative solve): Compute the previous-frame strains from the current (i.e. previous frame's converged) state.

$$\bar{\Gamma}_i \leftarrow \bar{q}_i \frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{l_i} q_i - e_3,$$
$$\bar{\Omega}_i \leftarrow \bar{q}_i \, q_{i+1}.$$

Orientation update (replacing the inner loop of Algorithm 2):

$$\begin{aligned}
\mathbf{v} &\leftarrow -2k_i^{ss}\, l_i^{-1}(\mathbf{x}_{i+1} - \mathbf{x}_i) && \text{(unchanged, eq. (5))}\\
\mathbf{r} &\leftarrow e_3 + \bar{\Gamma}_i && \text{(damping rest direction, eq. (11))}\\
\boldsymbol{\eta} &\leftarrow (1 + \alpha_{ss})\, e_3 + \alpha_{ss}\, \bar{\Gamma}_i && \text{(generalized axis, eq. (20))}\\
\mathbf{b} &\leftarrow \textstyle\sum \text{elastic bending terms} + \alpha_{bt} \textstyle\sum \text{damping bending terms} && \text{(eq. (21))}\\
\lambda &\leftarrow |\mathbf{v}||\boldsymbol{\eta}| + |\mathbf{b}| && \text{(approximate, eq. (30))}\\
q_i &\leftarrow \mathbf{v}\mathbf{b}\boldsymbol{\eta} + \lambda\mathbf{b} && \text{(generalized solve, eq. (28))}\\
q_i &\leftarrow q_i / |q_i|
\end{aligned}$$

Position update: Add the stretching damping force (eq. (33)) and Hessian (eq. (34)) to the existing stretching/shearing terms in the VBD block-diagonal solve.

The solver structure, parallelism, and convergence properties are all preserved. No additional persistent per-segment storage is required—$\bar{\Gamma}_i$ and $\bar{\Omega}_i$ are computed once at the start of each frame and reused across iterations.

References

[1] Jerry Hsu, Tongtong Wang, Kui Wu, and Cem Yuksel. Stable Cosserat Rods. In *SIGGRAPH Conference Papers '25*, August 2025. doi:10.1145/3721238.3730618.